Question Bank

Group Theory

Q 1 Prove that a group G is Abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all a and b in G.

Q 2 Let G be a group. Show that $Z(G) = \cap_{a\in G} C(a)$. [This means the intersection of *all* subgroups of the form $C(a)$.]

Q 3 . If a and b are distinct group elements, prove that either $a^2 \neq b^2$ or $a^3 \neq b^3$.

Q 4 . Find a cyclic subgroup of order 4 in $U(40)$.

Q 5 . Find a noncyclic subgroup of order 4 in $U(40)$.

Q 6 . Let $G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in Z \right\}$ under addition. Let $H = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G \mid a+b+c+d = 0 \right\}$. Prove that H is a subgroup of G. What if 0 is replaced by 1?

Q 7 Let $G = GL(2, \mathbf{R})$.

a. Find $C\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right)$.

b. Find $C\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$.

c. Find $Z(G)$.

Q 8 . Prove that in any group, an element and its inverse have the same order.

Q 9 If H is a subgroup of G, then by the *centralizer* $C(H)$ of H we mean the set $\{x \in G \mid xh = hx \text{ for all } h \in H\}$. Prove that $C(H)$ is a subgroup of G.

Q 10 Find an example of a noncyclic group, all of whose proper subgroups are cyclic.

Q 11 . In Z_{24}, list all generators for the subgroup of order 8. Let $G = \langle a \rangle$ and let $|a| = 24$. List all generators for the subgroup of order 8.

Q 12 Let G be a group and let $a \in G$. Prove that $\langle a^{-1} \rangle = \langle a \rangle$.

Q 13 In Z_{24}, find a generator for $\langle 21 \rangle \cap \langle 10 \rangle$. Suppose that $|a| = 24$. Find a generator for $\langle a^{21} \rangle \cap \langle a^{10} \rangle$. In general, what is a generator for the subgroup $\langle a^m \rangle \cap \langle a^n \rangle$?

Q 14 Prove that a group of order 3 must be cyclic.

Q 15 Let m and n be elements of the group Z. Find a generator for the group $\langle m \rangle \cap \langle n \rangle$.

Q 16 Suppose that a and b are group elements that commute and have orders m and n. If $\langle a \rangle \cap \langle b \rangle = \{e\}$, prove that the group contains an element whose order is the least common multiple of m and n. Show that this need not be true if a and b do not commute.

Q 17 Let a and b be elements of a group. If $|a| = 10$ and $|b| = 21$, show that $\langle a \rangle \cap \langle b \rangle = \{e\}$.

Q 18 Let a and b belong to a group. If $|a|$ and $|b|$ are relatively prime, show that $\langle a \rangle \cap \langle b \rangle = \{e\}$.

Q 19 Prove that $H = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \middle| \; n \in Z \right\}$ is a cyclic subgroup of $GL(2, \mathbf{R})$.

Q 20 Let

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 4 & 3 & 5 \end{bmatrix}.$$

Compute each of the following.

a. α^{-1}

b. $\beta\alpha$

c. $\alpha\beta$

Q 21 Let

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 6 \end{bmatrix} \text{and } \beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{bmatrix}.$$

Write α, β, and $\alpha\beta$ as

a. products of disjoint cycles;

b. products of 2-cycles.

Q 22 Let α and β belong to S_n. Prove that $\alpha\beta$ is even if and only if α and β are both even or both odd.

Q 23 Let $\beta = (123)(145)$. Write β^{99} in disjoint cycle form.

Q 24 Let $\beta = (1,3,5,7,9,8,6)(2,4,10)$. What is the smallest positive integer n for which $\beta^n = \beta^{-5}$?

Q 25 . Prove that S_n is non-Abelian for all $n \geq 3$.

Q 26 . Show that for $n \geq 3$, $Z(S_n) = \{\varepsilon\}$.

Q 27 Show that a permutation with odd order must be an even permutation.

Q 28 Why does the fact that the orders of the elements of A_4 are 1, 2, and 3 imply that $|Z(A_4)| = 1$?