Name of the Course	**: CBCS B.Sc. (H) Mathematics**
Unique Paper Code	**: 32351301_OC**
Name of the Paper	**: C 5 -Theory of Real Functions**
Semester	**: III**
Duration	**: 3 Hours**
Maximum Marks	**: 75**

Attempt any four questions. All questions carry equal marks.

1. Find the limit

$$\lim_{x\to 1} \frac{x^2 - x + 1}{x + 1}$$

and establish it by using ε-δ definition of the limit of a function.

Suppose that $\lim_{x\to c} f(x) = L$ where $L > 0$ and $\lim_{x\to c} g(x) = \infty$. Show that $\lim_{x\to c} f(x)\, g(x) = \infty$. If $L = 0$ then show by example that the conclusion may fail.

Let $x > 0$ and let $[x]$ denotes the greatest integer less than equal to x, then find

$$\lim_{x\to 0^+} \left\{x\left(\left[\frac{1}{x}\right] + \left[\frac{2}{x}\right] + \cdots + \left[\frac{7}{x}\right]\right)\right\}.$$

2. Use sequential criterion of continuity to prove that the function

$$f(x) = \begin{cases} \frac{\sin}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0, \end{cases}$$

is continuous at 0.

Let $S = \{x \in \mathbb{R} : f(x) = 0\}$ be the zero set of a function f. If $\{x_n\}$ is a sequence in S and $\lim_{n\to\infty} x_n = x$, Show that $x \in S$.

Let $I = [0, \pi/2]$ and let $f: I \to \mathbb{R}$ be defined by $f(x) = \sup\{x^2, \cos x\}, x \in I$. Show that there exists an absolute minimum point $x_0 \in I$ for f on I. Also show that x_0 is a solution of the equation $\cos x = x^2$.

3. Prove that a continuous real valued function defined on a closed and bounded interval is uniformly continuous therein.

Prove the inequality $\frac{x-1}{x} < \log x < x - 1$ for $x > 1$, by using mean valve theorem.

Show that the function $f(x) = \frac{1}{1+x^2}, x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

4. Suppose that f is a real valued function on $\mathbb{R}$ and that $f(a).f(b) < 0$ for some $a, b \in \mathbb{R}$. Prove that there exists x between a and b such that $f(x) = 0$.

Show that a continuous function $f: [0,1] \to [0,1]$, has a fixed point.

State and prove the chain rule of differentiation and use it differentiate the function $\sin\left(\sqrt{1 + \cos 2x}\right)$.

5. If f is continuous in $[a, b]$and differentiable in (a, b) then prove that there exists at least one $c \in (a, b)$ such that

$$\frac{f'(c)}{3c^2} = \frac{f(b)-f(a)}{b^3 - a^3}.$$

Prove that $\sin^2\theta < \theta.\sin(\sin\theta)$ for $0 < \theta < (\pi/2)$.

Determine the interval in which the function $f(x) = e^{\sqrt{x}}$ is convex.

6. Use Taylor's theorem to approximate sin(0.4) by fourth degree polynomial and determine the accuracy of the approximation.

Obtain the Maclaurin series expansion of the function $\cos^2 2x$.

Show that $e^{\pi} > \pi^{e}$.